USEFUL DERIVATIVES FOR GEODETIC TRANSFORMATIONS

R.E. Deakin[1] and M. N. Hunter [2]
[1] BONBEACH, VIC, 3196, Australia; [2] MARIBYRNONG, VIC, 3032, Australia.
Email: randm.deakin@gmail.com

Version 2: 01-Dec-2014

Abstract

Transformations of coordinates are common geodetic operations. For example, networks of GPS measurements are processed in X,Y,Z Cartesian coordinates but locations of network stations are usually transformed to geodetic coordinates ϕ,λ,h (latitude, longitude and height) related to the reference ellipsoid. And ϕ,λ may be transformed to E,N grid coordinates on a map projection. Such transformations are shown symbolically as $(X,Y,Z) \Leftrightarrow (\phi,\lambda,h)$ and $(\phi,\lambda) \Leftrightarrow (E,N)$ where $\Leftrightarrow$ represents sets of equations linking coordinates.

If the precisions of coordinates are known in one system, then precisions in the transformed system can be evaluated by *propagation of variances* expressed mathematically as a sequence of matrix operations. One of the matrices involved is the Jacobian matrix of first-order partial derivatives, and this paper gives the derivation of various partial derivatives as well as examples of their application.

Introduction

Something to be added here

Nomenclature

ε	eccentricity of ellipsoid
ε'	2nd eccentricity of ellipsoid
λ	longitude
λ_0	longitude of central meridian
ν	radius of curvature of ellipsoid in prime vertical plane
ρ	radius of curvature of ellipsoid in prime vertical plane
σ_x	standard deviation of random variable *x*
σ_x^2	variance of random variable *x*
σ_{xy}	covariance between random variables *x* and *y*
ϕ	latitude
ω	longitude difference: $\omega = \lambda - \lambda_0$
a	semi-major axis of ellipsoid
b	semi-minor axis of ellipsoid
c	polar radius of ellipsoid
E	east grid coordinate
E_0	false origin offset
f	flattening of ellipsoid
h	ellipsoidal height
m_0	central meridian scale factor
N	north grid coordinate
N_0	false origin offset
n	3rd flattening of ellipsoid
p	perpendicular distance from rotational axis of ellipsoid
R	radius of spherical earth
u	transverse Mercator coordinate (north)
V	latitude function
v	transverse Mercator coordinate (east)
W	latitude function
X	Cartesian coordinate
Y	Cartesian coordinate
Z	Cartesian coordinate

Ellipsoid and coordinates

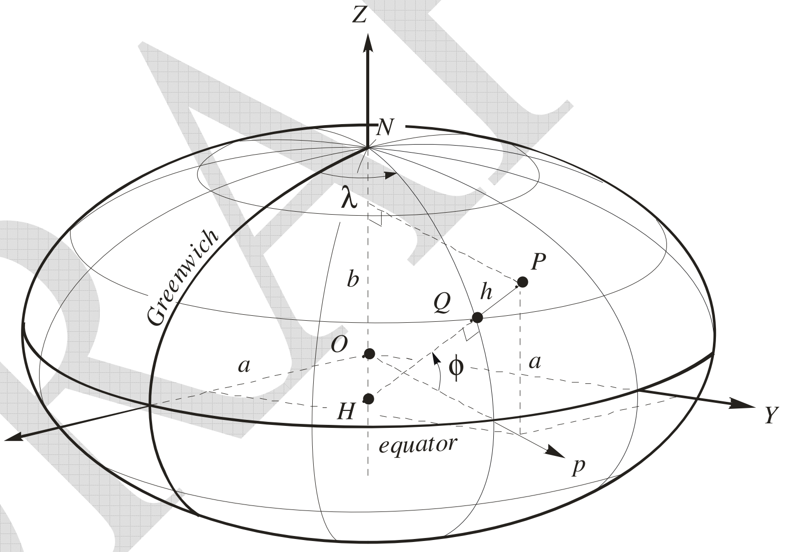

Figure 1 The ellipsoid and coordinates

Figure 1 shows a point *P* related to the centre *O* of an ellipsoid. In geodesy, the ellipsoid is a surface of revolution created by rotating an ellipse (whose semi-axes lengths are *a* and *b* and $a > b$) about its minor axis. The ellipsoid is the mathematical surface that idealizes the irregular shape of the earth and it has the following geometrical constants (Deakin and Hunter 2013):

flattening
$$f = \frac{a-b}{a} \tag{1}$$

eccentricity
$$\varepsilon = \sqrt{\frac{a^2-b^2}{a^2}} \tag{2}$$

2nd eccentricity
$$\varepsilon' = \sqrt{\frac{a^2-b^2}{b^2}} \tag{3}$$

3rd flattening $$n = \frac{a-b}{a+b} \tag{4}$$

polar radius $$c = \frac{a^2}{b} \tag{5}$$

These geometric constants are inter-related as follows

$$\frac{b}{a} = 1 - f = \sqrt{1-\varepsilon^2} = \frac{1}{\sqrt{1+\varepsilon'^2}} = \frac{1-n}{1+n} = \frac{a}{c} \tag{6}$$

$$\varepsilon^2 = \frac{\varepsilon'^2}{1+\varepsilon'^2} = f\left(2-f\right) = \frac{4n}{\left(1+n\right)^2} \tag{7}$$

$$\varepsilon'^2 = \frac{\varepsilon^2}{1-\varepsilon^2} = \frac{f\left(2-f\right)}{\left(1-f\right)^2} = \frac{4n}{\left(1-n\right)^2} \tag{8}$$

$$n = \frac{f}{2-f} = \frac{1-\sqrt{1-\varepsilon^2}}{1+\sqrt{1-\varepsilon^2}} \tag{9}$$

The ellipsoid radii of curvature ρ (meridian plane) and ν (prime vertical plane) at a point whose latitude is ϕ are

$$\rho = \frac{a\left(1-\varepsilon^2\right)}{\left(1-\varepsilon^2\sin^2\phi\right)^{3/2}} = \frac{a\left(1-\varepsilon^2\right)}{W^3} = \frac{c}{V^3} \tag{10}$$

$$\nu = \frac{a}{\left(1-\varepsilon^2\sin^2\phi\right)^{1/2}} = \frac{a}{W} = \frac{c}{V} \tag{11}$$

where the latitude functions *V* and *W* are defined as

$$W^2 = 1-\varepsilon^2\sin^2\phi \qquad V^2 = 1+\varepsilon'^2\cos^2\phi = \frac{1+n^2+2n\cos 2\phi}{\left(1-n\right)^2} = \frac{W^2}{1-\varepsilon^2} \tag{12}$$

The minor axis of the ellipsoid is assumed to be coincident with (or parallel to) the earth's rotational axis. The normal to the ellipsoid passing through *P* intersects the ellipsoid at *Q,* passes through the equatorial plane and intersects the axis of revolution at *H*. The distance $h = QH$ is the ellipsoidal height. The distance $\nu = QH$ is the radius of curvature of the ellipsoid in the prime-vertical plane. The angle ϕ between the normal through *P* and the equatorial plane is the latitude of *P*. Latitudes are measured 0° to 90° positive north and negative south of the equator. The plane containing *P, H* and *N* is the meridian plane of *P* and the angle λ between this plane and a reference meridian (Greenwich) is the longitude of *P*. Longitudes are measured 0° to 180° positive east and negative west of Greenwich. ϕ_P, λ_P, h_P are geographic coordinates of *P*.

Figure 1 shows an *X,Y,Z* Cartesian reference frame with an origin at the centre of the ellipsoid *O*. The *Z*-axis passes through *N,* the north pole of the ellipsoid; the *X-Z* plane is the reference plane for longitudes (Greenwich) and the *X-Y* plane is the equatorial plane of the ellipsoid. The *X*-axis passes through the intersection of the Greenwich meridian and the equator and the *Y*-axis is advanced 90° eastwards around the equator from the *X*-axis. X_P, Y_P, Z_P are Cartesian coordinates of *P*.

Propagation of Variances

Suppose that $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T$ and $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^T$ are vectors of random variables that are related through a function *f* thus:

$$\mathbf{y} = f\left(\mathbf{x}\right) \tag{13}$$

Then the general law of propagation of variances (Mikhail 1976) can be expressed as

$$\mathbf{\Sigma}_{yy} = \mathbf{J}\,\mathbf{\Sigma}_{xx}\mathbf{J}^T \tag{14}$$

Where $\mathbf{\Sigma}_{xx}$ and $\mathbf{\Sigma}_{yy}$ are square and symmetric variance matrices

$$\Sigma_{xx} = \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_m} \\ & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_m} \\ & & \ddots & \vdots \\ & & & \sigma_{x_m}^2 \end{bmatrix} \qquad \Sigma_{yy} = \begin{bmatrix} \sigma_{y_1}^2 & \sigma_{y_1y_2} & \cdots & \sigma_{y_1y_m} \\ & \sigma_{y_2}^2 & \cdots & \sigma_{y_2y_m} \\ & & \ddots & \vdots \\ & & & \sigma_{y_m}^2 \end{bmatrix} \tag{15}$$

Where the diagonal elements $\sigma_{x_1}^2, \sigma_{x_2}^2, \ldots \sigma_{x_m}^2$; $\sigma_{y_1}^2, \sigma_{y_2}^2, \ldots \sigma_{y_n}^2$ are variances and the off-diagonal elements $\sigma_{x_1x_2}$, $\sigma_{x_1x_2}, \ldots$ are covariances and $\mathbf{J}$ is a Jacobian matrix of partial derivatives given by

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \cdots & \dfrac{\partial y_1}{\partial x_m} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \cdots & \dfrac{\partial y_2}{\partial x_m} \\ \vdots & & & \vdots \\ \dfrac{\partial y_n}{\partial x_1} & \dfrac{\partial y_n}{\partial x_2} & \cdots & \dfrac{\partial y_n}{\partial x_m} \end{bmatrix} \tag{16}$$

As an example, consider a horizontal line of measured length *l* and measured bearing β where the standard deviations of length and bearing are σ_l and σ_β. The components *E* and *N* of the line are $E = l\sin\beta$ and $N = l\cos\beta$ which are computed quantities. What are the standard deviations of *E* and *N*?

Let $\mathbf{y} = \begin{bmatrix} E & N \end{bmatrix}^T$ and $\mathbf{x} = \begin{bmatrix} l & \beta \end{bmatrix}^T$ then $\Sigma_{xx} = \begin{bmatrix} \sigma_l^2 & 0 \\ 0 & \sigma_\beta^2 \end{bmatrix}$

noting that if the measurements *l* and β are independent then covariance $\sigma_{l\beta} = \sigma_{\beta l} = 0$ and

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial E}{\partial l} & \dfrac{\partial E}{\partial \beta} \\ \dfrac{\partial N}{\partial l} & \dfrac{\partial N}{\partial \beta} \end{bmatrix} = \begin{bmatrix} \sin\beta & l\cos\beta \\ \cos\beta & -l\sin\beta \end{bmatrix}$$

An application of (14) gives the variances and covariances of the computed quantities as

$$\mathbf{\Sigma}_{yy} = \begin{bmatrix} \sigma_E^2 & \sigma_{EN} \\ \sigma_{EN} & \sigma_N^2 \end{bmatrix} = \begin{bmatrix} \sin\beta & l\cos\beta \\ \cos\beta & -l\sin\beta \end{bmatrix} \begin{bmatrix} \sigma_l^2 & 0 \\ 0 & \sigma_\beta^2 \end{bmatrix} \begin{bmatrix} \sin\beta & \cos\beta \\ l\cos\beta & -l\sin\beta \end{bmatrix}$$

Geographic to Cartesian Transformations $(\phi,\lambda,h)\Rightarrow(X,Y,Z)$

where $\Rightarrow$ represents a set of equations that enable the transformation of geographical coordinates ϕ,λ,h to Cartesian coordinates *X,Y,Z.*

The Cartesian equations of a point related to the centre of an ellipsoid are (Deakin & Hunter 2013)

$$X=(\nu+h)\cos\phi\cos\lambda \tag{17}$$

$$Y=(\nu+h)\cos\phi\sin\lambda \tag{18}$$

$$Z=\left(\nu\left(1-\varepsilon^2\right)+h\right)\sin\phi \tag{19}$$

If ϕ,λ,h are regarded as random variables with variances $\sigma_\phi^2,\ \sigma_\lambda^2,\ \sigma_h^2$ and covariances $\sigma_{\phi\lambda}=\sigma_{\lambda\phi},\sigma_{\phi h}=\sigma_{h\phi},\sigma_{\lambda h}=\sigma_{h\lambda}$ then the variances and covariances of the computed quantities *X,Y,Z* can be evaluated from (14) where $\mathbf{y}=\begin{bmatrix}X & Y & Z\end{bmatrix}^T$, $\mathbf{x}=\begin{bmatrix}\phi & \lambda & h\end{bmatrix}^T$ and

$$\mathbf{\Sigma}_{XYZ}=\mathbf{J}\,\mathbf{\Sigma}_{\phi\lambda h}\,\mathbf{J}^T \tag{20}$$

where $$\mathbf{\Sigma}_{XYZ}=\begin{bmatrix}\sigma_X^2 & \sigma_{XY} & \sigma_{XZ}\\ \sigma_{XY} & \sigma_Y^2 & \sigma_{YZ}\\ \sigma_{XZ} & \sigma_{YZ} & \sigma_Z^2\end{bmatrix};\quad \mathbf{J}=\begin{bmatrix}\frac{\partial X}{\partial\phi} & \frac{\partial X}{\partial\lambda} & \frac{\partial X}{\partial h}\\ \frac{\partial Y}{\partial\phi} & \frac{\partial Y}{\partial\lambda} & \frac{\partial Y}{\partial h}\\ \frac{\partial Z}{\partial\phi} & \frac{\partial Z}{\partial\lambda} & \frac{\partial Z}{\partial h}\end{bmatrix};\quad \mathbf{\Sigma}_{\phi\lambda h}=\begin{bmatrix}\sigma_\phi^2 & \sigma_{\phi\lambda} & \sigma_{\phi h}\\ \sigma_{\lambda\phi} & \sigma_\lambda^2 & \sigma_{\lambda h}\\ \sigma_{h\phi} & \sigma_{h\lambda} & \sigma_h^2\end{bmatrix}$$

Equation (20) is given in Harvey (1986) and the partial derivatives in the Jacobian matrix **J** are derived in the following manner.

Substituting (11) and (12) into (17), (18) and (19) gives the *X,Y,Z* coordinates as functions of *a, W,* ϕ, λ and *h* as

$$X=\frac{a}{W}\cos\phi\cos\lambda+h\cos\phi\cos\lambda \tag{21}$$

$$Y=\frac{a}{W}\cos\phi\sin\lambda+h\cos\phi\sin\lambda \tag{22}$$

$$Z=\frac{a}{W}\left(1-\varepsilon^2\right)\sin\phi+h\sin\phi \tag{23}$$

Derivatives $\frac{\partial X}{\partial\phi},\frac{\partial Y}{\partial\phi},\frac{\partial Z}{\partial\phi}$

Using (10), (11), (12) and (21)

$$\begin{aligned}\frac{\partial X}{\partial\phi}&=-\frac{a}{W}\sin\phi\cos\lambda-\frac{a}{W^2}\frac{dW}{d\phi}\cos\phi\cos\lambda-h\sin\phi\cos\lambda\\ &=-\frac{a}{W}\sin\phi\cos\lambda-\frac{a}{W^2}\left(-\frac{\varepsilon^2\sin\phi\cos\phi}{W}\right)\cos\phi\cos\lambda-h\sin\phi\cos\lambda\\ &=-\nu\sin\phi\cos\lambda\left(1-\frac{\varepsilon^2\cos^2\phi}{W^2}\right)-h\sin\phi\cos\lambda\end{aligned}$$

Noting that $1-\frac{\varepsilon^2\cos^2\phi}{W^2}=\frac{1-\varepsilon^2\sin^2\phi-\varepsilon^2\cos^2\phi}{W^2}=\frac{1-\varepsilon^2\left(\sin^2\phi+\cos^2\phi\right)}{W^2}=\frac{1-\varepsilon^2}{W^2}=\frac{\rho}{\nu}$ then

$$\frac{\partial X}{\partial\phi}=-(\rho+h)\sin\phi\cos\lambda \tag{24}$$

Similarly, using (22)

$$\frac{\partial Y}{\partial \phi} = -(\rho + h)\sin\phi \sin\lambda \tag{25}$$

Using (10), (12) and (23)

$$\begin{aligned}
\frac{\partial Z}{\partial \phi} &= \frac{a}{W}\left(1-\varepsilon^2\right)\cos\phi - \frac{a}{W^2}\left(1-\varepsilon^2\right)\sin\phi \frac{dW}{d\phi} + h\cos\phi \\
&= \frac{a}{W}\left(1-\varepsilon^2\right)\cos\phi - \frac{a}{W^2}\left(1-\varepsilon^2\right)\sin\phi\left(-\frac{\varepsilon^2 \sin\phi\cos\phi}{W}\right) + h\cos\phi \\
&= \nu\left(1-\varepsilon^2\right)\left(1+\frac{\varepsilon^2\sin^2\phi}{W^2}\right)\cos\phi + h\cos\phi
\end{aligned}$$

and

$$\frac{\partial Z}{\partial \phi} = (\rho + h)\cos\phi \tag{26}$$

Derivatives $\frac{\partial X}{\partial \lambda}, \frac{\partial Y}{\partial \lambda}, \frac{\partial Z}{\partial \lambda}$

Using (21), (11) and (12)

$$\frac{\partial X}{\partial \lambda} = -\frac{a}{W}\cos\phi\sin\lambda - h\cos\phi\sin\lambda = -(\nu + h)\cos\phi\sin\lambda \tag{27}$$

Similarly

$$\frac{\partial Y}{\partial \lambda} = (\nu + h)\cos\phi\cos\lambda \tag{28}$$

And, since Z is independent of λ

$$\frac{\partial Z}{\partial \lambda} = 0 \tag{29}$$

Derivatives $\frac{\partial X}{\partial h}, \frac{\partial Y}{\partial h}, \frac{\partial Z}{\partial h}$

Using (21)

$$\frac{\partial X}{\partial h} = \cos\phi\cos\lambda \tag{30}$$

Similarly

$$\frac{\partial Y}{\partial h} = \cos\phi\sin\lambda \tag{31}$$

and

$$\frac{\partial Z}{\partial h} = \sin\phi \tag{32}$$

The Jacobian matrix **J** for propagation of variances in Geographic to Cartesian transformation is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial X}{\partial \phi} & \frac{\partial X}{\partial \lambda} & \frac{\partial X}{\partial h} \\ \frac{\partial Y}{\partial \phi} & \frac{\partial Y}{\partial \lambda} & \frac{\partial Y}{\partial h} \\ \frac{\partial Z}{\partial \phi} & \frac{\partial Z}{\partial \lambda} & \frac{\partial Z}{\partial h} \end{bmatrix} = \begin{bmatrix} -(\rho+h)\sin\phi\cos\lambda & -(\nu+h)\cos\phi\sin\lambda & \cos\phi\cos\lambda \\ -(\rho+h)\sin\phi\sin\lambda & (\nu+h)\cos\phi\cos\lambda & \cos\phi\sin\lambda \\ (\rho+h)\cos\phi & 0 & \sin\phi \end{bmatrix} \tag{33}$$

This is equivalent to Harvey (1986, eq. 5, p. 110). The partial derivatives in **J** are also given in Soler (1976, eq. 3.2-8, p. 11).

Cartesian to Geographic Transformations $(X,Y,Z) \Rightarrow (\phi,\lambda,h)$

Where $\Rightarrow$ represents the set of equations that enable the transformation of Cartesian coordinates *X,Y,Z* to geographical coordinates ϕ,λ,h .

The geographic coordinates of a point whose *X,Y,Z* coordinates related to the centre of an ellipsoid are known can be evaluated from (Deakin & Hunter 2013)

$$p\tan\phi = Z + \nu\varepsilon^2\sin\phi \tag{34}$$

$$p^2 = X^2 + Y^2 \tag{35}$$

$$\nu^2\left(1-\varepsilon^2\sin^2\phi\right) = a^2 \tag{36}$$

$$Y = X\tan\lambda \tag{37}$$

$$h = p\sec\phi - \nu \tag{38}$$

If *X,Y,Z* are regarded as random variables with variances $\sigma_X^2, \sigma_Y^2, \sigma_Z^2$ and covariances $\sigma_{XY} = \sigma_{YX}, \sigma_{XZ} = \sigma_{ZX}, \sigma_{YZ} = \sigma_{ZY}$ then the variances and covariances of the computed quantities ϕ,λ,h can be evaluated from (14) where $\mathbf{y} = [\phi \quad \lambda \quad h]^T$, $\mathbf{x} = [X \quad Y \quad Z]^T$ and

$$\mathbf{\Sigma}_{\phi\lambda h} = \mathbf{J}^* \mathbf{\Sigma}_{XYZ} \mathbf{J}^{*T} \tag{39}$$

where $\mathbf{\Sigma}_{\phi\lambda h} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} & \sigma_{\phi h} \\ \sigma_{\lambda\phi} & \sigma_\lambda^2 & \sigma_{\lambda h} \\ \sigma_{h\phi} & \sigma_{h\lambda} & \sigma_h^2 \end{bmatrix}$; $\mathbf{J}^* = \begin{bmatrix} \frac{\partial \phi}{\partial X} & \frac{\partial \phi}{\partial Y} & \frac{\partial \phi}{\partial Z} \\ \frac{\partial \lambda}{\partial X} & \frac{\partial \lambda}{\partial Y} & \frac{\partial \lambda}{\partial Z} \\ \frac{\partial h}{\partial X} & \frac{\partial h}{\partial Y} & \frac{\partial h}{\partial Z} \end{bmatrix}$; $\mathbf{\Sigma}_{XYZ} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{YX} & \sigma_Y^2 & \sigma_{YZ} \\ \sigma_{ZX} & \sigma_{ZY} & \sigma_Z^2 \end{bmatrix}$

The partial derivatives in the Jacobian matrix $\mathbf{J}^*$ are derived in the following manner.

Derivatives $\frac{\partial \phi}{\partial X}, \frac{\partial \phi}{\partial Y}, \frac{\partial \phi}{\partial Z}$

Partially differentiate (34) with respect to *X* keeping *Y* and *Z* fixed gives

$$\frac{\partial p}{\partial X}\tan\phi + p\sec^2\phi\frac{\partial \phi}{\partial X} = \frac{\partial \nu}{\partial X}\varepsilon^2\sin\phi + \nu\varepsilon^2\cos\phi\frac{\partial \phi}{\partial X} \tag{40}$$

From (36) $\nu W = a$, so that $\frac{\partial \nu}{\partial X}W + \nu\frac{\partial W}{\partial X} = 0$ from which $\frac{\partial \nu}{\partial X}W^2 = -\nu W\frac{\partial W}{\partial X}$

By (12) $W\frac{\partial W}{\partial X} = -\varepsilon^2\sin\phi\cos\phi\frac{\partial \phi}{\partial X}$, so that $\frac{\partial \nu}{\partial X}W^2 = \nu\varepsilon^2\sin\phi\cos\phi\frac{\partial \phi}{\partial X}$ and hence

$$\frac{\partial \nu}{\partial X} = \frac{\nu^3 \varepsilon^2}{a^2} \sin\phi \cos\phi \frac{\partial \phi}{\partial X} \tag{41}$$

Differentiate (35) with respect to *X* to obtain $p\frac{\partial p}{\partial X} = X$ and so

$$\frac{\partial p}{\partial X} = \frac{X}{p} = \cos\lambda \tag{42}$$

Substituting (41) and (42) into (40) and re-arranging gives

$$\frac{\partial \phi}{\partial X} = \frac{\cos\lambda \tan\phi}{\frac{\nu^3 \varepsilon^2}{a^2}\cos\phi - p\sec^2\phi} \tag{43}$$

The numerator of (43) can be simplified as follows

$$\begin{aligned}\frac{\nu^3 \varepsilon^2}{a^2}\cos\phi - p\sec^2\phi &= \frac{a}{W^3}\varepsilon^2\cos\phi - \left(h + \frac{a}{W}\right)\sec\phi \\ &= \frac{\rho}{1-\varepsilon^2}\varepsilon^2\cos\phi + \frac{\rho}{1-\varepsilon^2}\left(1-\varepsilon^2\sin^2\phi\right)\sec\phi - h\sec\phi \\ &= \frac{\rho\sec\phi}{1-\varepsilon^2}\left(\varepsilon^2\cos^2\phi - 1 + \varepsilon^2\sin^2\phi\right) - h\sec\phi \\ &= \frac{-(\rho+h)}{\cos\phi}\end{aligned} \tag{44}$$

Using (44) in (43) gives

$$\frac{\partial \phi}{\partial X} = \frac{-\sin\phi\cos\lambda}{\rho+h} \tag{45}$$

Similarly

$$\frac{\partial \phi}{\partial Y} = \frac{\sin\lambda\tan\phi}{\frac{\nu^3 \varepsilon^2}{a^2}\cos\phi - p\sec^2\phi} = \frac{-\sin\phi\sin\lambda}{\rho+h} \tag{46}$$

Finally

$$\frac{\partial \phi}{\partial Z} = \frac{1}{p\sec^2\phi - \frac{\nu^3 \varepsilon^2}{a^2}\cos\phi} = \frac{\cos\phi}{\rho+h} \tag{47}$$

Derivatives $\frac{\partial \lambda}{\partial X}, \frac{\partial \lambda}{\partial Y}, \frac{\partial \lambda}{\partial Z}$

Partially differentiating (37) with respect to *X* gives

$$0 = \tan\lambda + X\sec^2\lambda \frac{\partial \lambda}{\partial X}$$

And so

$$\frac{\partial \lambda}{\partial X} = \frac{-\sin\lambda\cos\lambda}{X} = \frac{-\sin\lambda}{p} = \frac{-\sin\lambda}{(\nu+h)\cos\phi} \tag{48}$$

Similarly

$$\frac{\partial \lambda}{\partial Y} = \frac{\cos^2\lambda}{X} = \frac{\cos\lambda}{p} = \frac{\cos\lambda}{(\nu+h)\cos\phi} \tag{49}$$

Finally, partially differentiating (37) with respect to Z gives

$$\frac{\partial \lambda}{\partial Z} = 0 \tag{50}$$

Derivatives $\frac{\partial h}{\partial X}, \frac{\partial h}{\partial Y}, \frac{\partial h}{\partial Z}$

Differentiate (38) with respect to X to obtain

$$\frac{\partial h}{\partial X} = \frac{\partial p}{\partial X}\sec\phi + p\sec\phi\tan\phi\frac{\partial \phi}{\partial X} - \frac{\partial \nu}{\partial X}$$

And so

$$\frac{\partial h}{\partial X} = \cos\lambda\sec\phi + \frac{\cos\lambda\tan\phi}{\frac{\nu^3\varepsilon^2}{a^2}\cos\phi - p\sec^2\phi}\left(p\sec^2\phi - \frac{\nu^3\varepsilon^2}{a^2}\cos\phi\right)\sin\phi \tag{51}$$

Using (44) in (51) and simplifying gives

$$\frac{\partial h}{\partial X} = \cos\phi\cos\lambda \tag{52}$$

Similarly

$$\frac{\partial h}{\partial Y} = \sin\lambda\sec\phi + \frac{\sin\lambda\tan\phi}{\frac{\nu^3\varepsilon^2}{a^2}\cos\phi - p\sec^2\phi}\left(p\sec^2\phi - \frac{\nu^3\varepsilon^2}{a^2}\cos\phi\right)\sin\phi = \cos\phi\sin\lambda \tag{53}$$

And

$$\frac{\partial h}{\partial Z} = \frac{1}{p\sec^2\phi - \frac{\nu^3\varepsilon^2}{a^2}\cos\phi}\left(p\sec^2\phi - \frac{\nu^3\varepsilon^2}{a^2}\cos\phi\right)\sin\phi = \sin\phi \tag{54}$$

The Jacobian matrix $\mathbf{J}^*$ for propagation of variances in Cartesian to Geographic transformation is

$$\mathbf{J}^* = \begin{bmatrix} \frac{\partial \phi}{\partial X} & \frac{\partial \phi}{\partial Y} & \frac{\partial \phi}{\partial Z} \\ \frac{\partial \lambda}{\partial X} & \frac{\partial \lambda}{\partial Y} & \frac{\partial \lambda}{\partial Z} \\ \frac{\partial h}{\partial X} & \frac{\partial h}{\partial Y} & \frac{\partial h}{\partial Z} \end{bmatrix} = \begin{bmatrix} \frac{-\sin\phi\cos\lambda}{\rho + h} & \frac{-\sin\phi\sin\lambda}{\rho + h} & \frac{\cos\phi}{\rho + h} \\ \frac{-\sin\lambda}{(\nu + h)\cos\phi} & \frac{\cos\lambda}{(\nu + h)\cos\phi} & 0 \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \tag{55}$$

The partial derivatives in $\mathbf{J}^*$ are given in Soler (1976, eq. 3.7-2, p. 20).

Harvey (1986, eq. 8, p. 111) has alternative expressions (some containing approximations) for the partial derivatives in $\mathbf{J}^*$. The interested reader with some algebra and inspiration from the derivations above should be able to obtain our expressions from Harvey's.

An alternative approach to determining the elements of the Jacobian matrix $\mathbf{J}^*$ follows from (20) and with some matrix algebra (noting that $\mathbf{\Sigma}_{\phi\lambda h}$ and $\mathbf{\Sigma}_{XYZ}$ are symmetric) we write

$$\begin{aligned} \mathbf{\Sigma}_{XYZ} &= \mathbf{J}\,\mathbf{\Sigma}_{\phi\lambda h}\,\mathbf{J}^T \\ \mathbf{J}^{-1}\,\mathbf{\Sigma}_{XYZ} &= \mathbf{\Sigma}_{\phi\lambda h}\,\mathbf{J}^T \\ \mathbf{\Sigma}_{XYZ}^T\left(\mathbf{J}^{-1}\right)^T &= \mathbf{J}\,\mathbf{\Sigma}_{\phi\lambda h}^T \\ \mathbf{J}^{-1}\,\mathbf{\Sigma}_{XYZ}\left(\mathbf{J}^{-1}\right)^T &= \mathbf{\Sigma}_{\phi\lambda h} \end{aligned} \tag{56}$$

And comparing (56) and (39) implies

$$\mathbf{J}^{*} = \mathbf{J}^{-1} \tag{57}$$

Following Deakin (2004) the inverse of **J** can be found by the method of *cofactors* and *adjoints* (Mikhail 1973, pp. 442-5).

For a 3×3 matrix $\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ the inverse $\mathbf{A}^{-1}$ is given by $\mathbf{A}^{-1}=\dfrac{adj\,\mathbf{A}}{|\mathbf{A}|}$ where $adj\,\mathbf{A}$ is the *adjoint* matrix and $|\mathbf{A}|$ is the *determinant* of **A**, a scalar quantity. Each element a_{kj} of **A** has a *minor* m_{kj} and a *cofactor* c_{kj}. The minor of each element is the determinant of the elements of **A** remaining after row *k* and column *j* are deleted, eg, $m_{11}=a_{22}a_{33}-a_{23}a_{32}$, $m_{22}=a_{11}a_{33}-a_{13}a_{31}$ and $m_{32}=a_{11}a_{23}-a_{13}a_{21}$. The cofactors $c_{kj}=(-1)^{k+j}\,m_{kj}$ form a matrix **C** whose transpose is the adjoint matrix. The determinant $|\mathbf{A}|=\sum_{j=1}^{j=3} a_{kj}c_{kj}$ and *k* = 1, 2, or 3.

The elements of the cofactor matrix of **J** are

$$\begin{aligned} c_{11} &= +\left\{\left[(\nu+h)\cos\phi\cos\lambda\right][\sin\phi]-[\cos\phi\sin\lambda][0]\right\} \\ &= (\nu+h)\sin\phi\cos\phi\cos\lambda \end{aligned}$$

$$\begin{aligned} c_{12} &= -\left\{\left[-(\rho+h)\sin\phi\sin\lambda\right][\sin\phi]-[\cos\phi\sin\lambda]\left[(\rho+h)\cos\phi\right]\right\} \\ &= -\left\{-(\rho+h)\sin\lambda\left(\sin^2\phi+\cos^2\phi\right)\right\} \\ &= (\rho+h)\sin\lambda \end{aligned}$$

$$\begin{aligned} c_{13} &= +\left\{\left[-(\rho+h)\sin\phi\sin\lambda\right][0]-\left[(\nu+h)\cos\phi\cos\lambda\right]\left[(\rho+h)\cos\phi\right]\right\} \\ &= -(\nu+h)\cos\phi\cos\lambda(\rho+h)\cos\phi \end{aligned}$$

$$\begin{aligned} c_{21} &= -\left\{\left[-(\nu+h)\cos\phi\sin\lambda\right][\sin\phi]-[\cos\phi\cos\lambda][0]\right\} \\ &= (\nu+h)\sin\phi\cos\phi\sin\lambda \end{aligned}$$

$$\begin{aligned} c_{22} &= +\left\{\left[-(\rho+h)\sin\phi\cos\lambda\right][\sin\phi]-[\cos\phi\cos\lambda]\left[(\rho+h)\cos\phi\right]\right\} \\ &= -(\rho+h)\cos\lambda\left(\sin^2\phi+\cos^2\phi\right) \\ &= -(\rho+h)\cos\lambda \end{aligned}$$

$$\begin{aligned} c_{23} &= -\left\{\left[-(\rho+h)\sin\phi\cos\lambda\right][0]-\left[-(\nu+h)\cos\phi\sin\lambda\right]\left[(\rho+h)\cos\phi\right]\right\} \\ &= -(\nu+h)\cos\phi\sin\lambda(\rho+h)\cos\phi \end{aligned}$$

$$\begin{aligned} c_{31} &= +\left\{\left[-(\nu+h)\cos\phi\sin\lambda\right][\cos\phi\sin\lambda]-[\cos\phi\cos\lambda]\left[(\nu+h)\cos\phi\cos\lambda\right]\right\} \\ &= -(\nu+h)\cos^2\phi\left(\sin^2\lambda+\cos^2\lambda\right) \\ &= -(\nu+h)\cos^2\phi \end{aligned}$$

$$\begin{aligned} c_{32} &= -\left\{\left[-(\rho+h)\sin\phi\cos\lambda\right][\cos\phi\sin\lambda]-[\cos\phi\cos\lambda]\left[-(\rho+h)\sin\phi\sin\lambda\right]\right\} \\ &= -\left\{-(\rho+h)\cos\phi\cos\lambda(\sin\phi\sin\lambda-\sin\phi\sin\lambda)\right\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} c_{33} &= +\left\{\left[-(\rho+h)\sin\phi\cos\lambda\right]\left[(\nu+h)\cos\phi\cos\lambda\right]-\left[-(\nu+h)\cos\phi\sin\lambda\right]\left[-(\rho+h)\sin\phi\sin\lambda\right]\right\} \\ &= +\left\{-(\rho+h)\sin\phi\cos\phi(\nu+h)\left(\cos^2\lambda+\sin^2\lambda\right)\right\} \\ &= -(\rho+h)\sin\phi(\nu+h)\cos\phi \end{aligned}$$

The determinant $|\mathbf{J}|$ is given by

$$
\begin{aligned}
|\mathbf{J}| &= j_{31}c_{31} + j_{32}c_{32} + j_{33}c_{33} \\
&= (\rho+h)\cos\phi\left(-(\nu+h)\cos^2\phi\right)+0+\sin\phi\left(-(\rho+h)\sin\phi(\nu+h)\cos\phi\right) \\
&= -(\nu+h)\cos\phi\left((\rho+h)\cos^2\phi+(\rho+h)\sin^2\phi\right) \\
&= -(\nu+h)\cos\phi(\rho+h)
\end{aligned}
$$

The inverse $\mathbf{J}^{-1} = \dfrac{adj\,\mathbf{J}}{|\mathbf{J}|} = \dfrac{\mathbf{C}^T}{|\mathbf{J}|}$ and

$$
\mathbf{J}^* = \mathbf{J}^{-1} = \begin{bmatrix} \frac{\partial\phi}{\partial X} & \frac{\partial\phi}{\partial Y} & \frac{\partial\phi}{\partial Z} \\ \frac{\partial\lambda}{\partial X} & \frac{\partial\lambda}{\partial Y} & \frac{\partial\lambda}{\partial Z} \\ \frac{\partial h}{\partial X} & \frac{\partial h}{\partial Y} & \frac{\partial h}{\partial Z} \end{bmatrix} = \begin{bmatrix} \frac{-\sin\phi\cos\lambda}{\rho+h} & \frac{-\sin\phi\sin\lambda}{\rho+h} & \frac{\cos\phi}{\rho+h} \\ \frac{-\sin\lambda}{(\nu+h)\cos\phi} & \frac{\cos\lambda}{(\nu+h)\cos\phi} & 0 \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \tag{58}
$$

Geographic to Transverse Mercator Grid Transformation $(\phi,\lambda)\Rightarrow(E,N)$

Where $\Rightarrow$ represents the set of equations that enable the transformation of geographical coordinates ϕ,λ to Transverse Mercator (TM) Grid coordinates *E,N*.

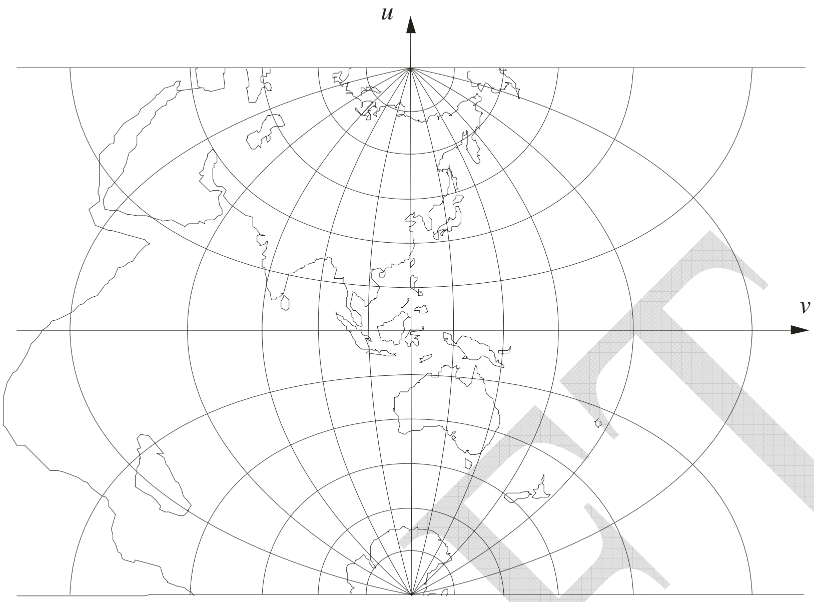

Figure 2 Transverse Mercator projection ($\lambda_0 = 120^\circ$ East)

The TM projection coordinates *u,v* of a point whose latitude and longitude are ϕ,λ on a spherical earth of radius *R* are (Deakin *et al.* 2012)

$$u = R\arctan\left(\frac{\tan\phi}{\cos\omega}\right) \tag{59}$$

$$v = \frac{R}{2}\ln\left(\frac{1+\cos\phi\sin\omega}{1-\cos\phi\sin\omega}\right) = \frac{R}{2}\ln(1+\cos\phi\sin\omega) - \frac{R}{2}\ln(1-\cos\phi\sin\omega) \tag{60}$$

where

$$\omega = \lambda - \lambda_0 \tag{61}$$

and λ_0 is the longitude of a chosen central meridian.

In Figure 2 the *u*-axis points north and lies on the central meridian of the projection. The *v*-axis points east and lies on the equator. The *u,v* coordinate origin is the *true origin* of the projection. The TM projection is ideally suited to display northern and southern zones of the earth with large latitude extent and small longitude extent.

For convenience a TM Grid (*E,N*) is superimposed over a region of interest with a *false origin* offset from the true origin and

$$E = v + E_0 \tag{62}$$

$$N = u + N_0 \tag{63}$$

For northern zones the false origin is a distance E_0 west of the central meridian and N_0 north of the equator. For southern zones the false origin is E_0 west and N_0 south of the equator.

If ϕ, λ are regarded as random variables with variances $\sigma_\phi^2, \sigma_\lambda^2$ and covariances $\sigma_{\phi\lambda} = \sigma_{\lambda\phi}$ then the variances and covariances of the computed quantities *E,N* can be evaluated from (14) where $\mathbf{y} = \begin{bmatrix} E & N \end{bmatrix}^T$, $\mathbf{x} = \begin{bmatrix} \phi & \lambda \end{bmatrix}^T$ and

$$\mathbf{\Sigma}_{EN} = \mathbf{J}\,\mathbf{\Sigma}_{\phi\lambda}\,\mathbf{J}^T \tag{64}$$

where
$$\mathbf{\Sigma}_{EN} = \begin{bmatrix} \sigma_E^2 & \sigma_{EN} \\ \sigma_{EN} & \sigma_N^2 \end{bmatrix}; \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial E}{\partial \phi} & \dfrac{\partial E}{\partial \lambda} \\ \dfrac{\partial N}{\partial \phi} & \dfrac{\partial N}{\partial \lambda} \end{bmatrix}; \quad \mathbf{\Sigma}_{\phi\lambda} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} \\ \sigma_{\lambda\phi} & \sigma_\lambda^2 \end{bmatrix}$$

The partial derivatives in the Jacobian **J** are derived in the following manner

Derivatives $\dfrac{\partial E}{\partial \phi}, \dfrac{\partial E}{\partial \lambda}$

Because of (61) and (62) and since λ_0 and E_0 are constants $\dfrac{\partial E}{\partial \phi} = \dfrac{\partial v}{\partial \phi}$, $\dfrac{\partial E}{\partial \lambda} = \dfrac{\partial v}{\partial \omega}$.

Differentiate (60) with respect to ϕ keeping ω fixed yields

$$\begin{aligned} \frac{\partial v}{\partial \phi} &= \tfrac{1}{2} R \frac{-\sin\phi \sin\omega}{1 + \cos\phi \sin\omega} - \tfrac{1}{2} R \frac{\sin\phi \sin\omega}{1 - \cos\phi \sin\omega} \\ &= -\tfrac{1}{2} R \sin\phi \sin\omega \left(\frac{1 - \cos\phi \sin\omega + 1 + \cos\phi \sin\omega}{(1 + \cos\phi \sin\omega)(1 - \cos\phi \sin\omega)} \right) \\ &= \frac{-R \sin\phi \sin\omega}{1 - \cos^2\phi \sin^2\omega} \end{aligned}$$

And so

$$\frac{\partial E}{\partial \phi} = \frac{-R \sin\phi \sin\omega}{1 - \cos^2\phi \sin^2\omega} \tag{65}$$

Similarly

$$\frac{\partial E}{\partial \lambda} = \frac{R \cos\phi \cos\omega}{1 - \cos^2\phi \sin^2\omega} \tag{66}$$

Derivatives $\dfrac{\partial N}{\partial \phi}, \dfrac{\partial N}{\partial \lambda}$

Because of (63) and since N_0 is a constant $\dfrac{\partial N}{\partial \phi} = \dfrac{\partial u}{\partial \phi}$, $\dfrac{\partial N}{\partial \lambda} = \dfrac{\partial u}{\partial \omega}$.

Differentiate (59) with respect to ϕ gives

$$\frac{\partial u}{\partial \phi} = \frac{R}{1 + \left(\dfrac{\tan\phi}{\cos\omega} \right)^2} \frac{\sec^2\phi}{\cos\omega} = \frac{R\cos\omega}{\cos^2\phi \cos^2\omega + \sin^2\phi} = \frac{R\cos\omega}{1 - \cos^2\phi \sin^2\omega}$$

So that

$$\frac{\partial N}{\partial \phi} = \frac{R\cos\omega}{1 - \cos^2\phi \sin^2\omega} \tag{67}$$

Similarly

$$\frac{\partial N}{\partial \lambda} = \frac{R \sin\phi \cos\phi \sin\omega}{1 - \cos^2\phi \sin^2\omega} \tag{68}$$

The Jacobian matrix **J** for propagation of variances in Geographic to TM Grid transformation is

$$\mathbf{J}=\begin{bmatrix}\dfrac{\partial E}{\partial\phi} & \dfrac{\partial E}{\partial\lambda}\\ \dfrac{\partial N}{\partial\phi} & \dfrac{\partial N}{\partial\lambda}\end{bmatrix}=\frac{R}{1-\cos^2\phi\sin^2\omega}\begin{bmatrix}-\sin\phi\sin\omega & \cos\phi\cos\omega\\ \cos\omega & \sin\phi\cos\phi\sin\omega\end{bmatrix}\tag{69}$$

Transverse Mercator Grid to Geographic Transformation $(E,N)\Rightarrow(\phi,\lambda)$

Where $\Rightarrow$ represents the set of equations that enable the transformation from TM Grid coordinates *E,N* to geographical coordinates ϕ,λ.

If *E,N* are regarded as random variables with variances $\sigma_E^2,\ \sigma_N^2$ and covariances $\sigma_{EN}=\sigma_{NE}$ then the variances and covariances of the computed quantities ϕ,λ can be evaluated from (14) where $\mathbf{y}=\begin{bmatrix}\phi & \lambda\end{bmatrix}^T$, $\mathbf{x}=\begin{bmatrix}E & N\end{bmatrix}^T$ and

$$\mathbf{\Sigma}_{\phi\lambda}=\mathbf{J}\,\mathbf{\Sigma}_{EN}\,\mathbf{J}^T\tag{70}$$

where $$\mathbf{\Sigma}_{\phi\lambda}=\begin{bmatrix}\sigma_\phi^2 & \sigma_{\phi\lambda}\\ \sigma_{\lambda\phi} & \sigma_\lambda^2\end{bmatrix};\quad \mathbf{J}=\begin{bmatrix}\dfrac{\partial\phi}{\partial E} & \dfrac{\partial\phi}{\partial N}\\ \dfrac{\partial\lambda}{\partial E} & \dfrac{\partial\lambda}{\partial N}\end{bmatrix};\quad \mathbf{\Sigma}_{EN}=\begin{bmatrix}\sigma_E^2 & \sigma_{EN}\\ \sigma_{EN} & \sigma_N^2\end{bmatrix}$$

The partial derivatives in the Jacobian **J** are derived from (64) and with some matrix algebra (noting that $\mathbf{\Sigma}_{\phi\lambda}$ and $\mathbf{\Sigma}_{EN}$ are symmetric) in the following manner

$$\begin{aligned}\mathbf{\Sigma}_{EN}&=\mathbf{J}\,\mathbf{\Sigma}_{\phi\lambda}\,\mathbf{J}^T\\ \mathbf{J}^{-1}\,\mathbf{\Sigma}_{EN}&=\mathbf{\Sigma}_{\phi\lambda}\,\mathbf{J}^T\\ \mathbf{\Sigma}_{EN}^T\left(J^{-1}\right)^T&=\mathbf{J}\,\mathbf{\Sigma}_{\phi\lambda}^T\\ \mathbf{J}^{-1}\,\mathbf{\Sigma}_{EN}\left(\mathbf{J}^{-1}\right)^T&=\mathbf{\Sigma}_{\phi\lambda}\end{aligned}\tag{71}$$

Since **J** is 2×2 matrices we use the standard matrix result:

If $\mathbf{A}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ is non-singular, the inverse $\mathbf{A}^{-1}=\dfrac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}$

And, using a rule for matrix inverse: $(\alpha\mathbf{A})^{-1}=\dfrac{1}{\alpha}\mathbf{A}^{-1}$ (for α a scalar) we write **Error! Reference source not found.** as

$$\begin{aligned}\mathbf{J}^{-1}&=\left(\frac{R}{1-\cos^2\phi\sin^2\omega}\begin{bmatrix}-\sin\phi\sin\omega & \cos\phi\cos\omega\\ \cos\omega & \sin\phi\cos\phi\sin\omega\end{bmatrix}\right)^{-1}\\ &=\left(\frac{1-\cos^2\phi\sin^2\omega}{R}\right)\left(\frac{1}{-\sin^2\phi\cos\phi\sin^2\omega-\cos\phi\cos^2\omega}\right)\begin{bmatrix}\sin\phi\cos\phi\sin\omega & -\cos\phi\cos\omega\\ -\cos\omega & -\sin\phi\sin\omega\end{bmatrix}\end{aligned}$$

Noting that

$$\begin{aligned}-\sin^2\phi\cos\phi\sin^2\omega-\cos\phi\cos^2\omega&=-\cos\phi\left(\sin^2\phi\sin^2\omega+\cos^2\omega\right)\\ &=-\cos\phi\left(\left(1-\cos^2\phi\right)\sin^2\omega+\cos^2\omega\right)\\ &=-\cos\phi\left(1-\cos^2\phi\sin^2\omega\right)\end{aligned}$$

The inverse of the Jacobian matrix is

$$\mathbf{J}^{-1} = \begin{bmatrix} \dfrac{\partial \phi}{\partial E} & \dfrac{\partial \phi}{\partial N} \\ \dfrac{\partial \lambda}{\partial E} & \dfrac{\partial \lambda}{\partial N} \end{bmatrix} = \frac{1}{R} \left[\begin{array}{c|c} -\sin\phi \sin\omega & \cos\omega \\ \hline \sec\phi \cos\omega & \tan\phi \sin\omega \end{array} \right] \tag{72}$$

And the partial derivatives are

$$\frac{\partial \phi}{\partial E} = \frac{-\sin\phi \sin\omega}{R} \tag{73}$$

$$\frac{\partial \phi}{\partial N} = \frac{\cos\omega}{R} \tag{74}$$

$$\frac{\partial \lambda}{\partial E} = \frac{\cos\omega}{R\cos\phi} \tag{75}$$

$$\frac{\partial \lambda}{\partial N} = \frac{\sin\phi \sin\omega}{R\cos\phi} \tag{76}$$

Example 1

A recent GPS campaign in the Rotorua region New Zealand included observations at Mt. Ngongotaha. The processing software gave the following Cartesian coordinates:

Mt. Ngongotaha:

$$\begin{aligned} X &= -5013888.2149 \text{ m} & \sigma_X &= 0.0124 \text{ m} \\ Y &= 333204.0203 \text{ m} & \sigma_Y &= 0.0077 \text{ m} \\ Z &= -3916273.4839 \text{ m} & \sigma_Z &= 0.0097 \text{ m} \end{aligned}$$

with standard deviations/correlations:

	X(m)	Y(m)	Z(m)
X(m)	0.0124	–0.0922	0.9291
Y(m)	–0.0922	0.0077	–0.0871
Z(m)	0.9291	–0.0871	0.0097

(77)

Correlations $(\rho_{XY} = \rho_{YX}, \rho_{XZ} = \rho_{ZX}, \rho_{YZ} = \rho_{ZY})$ which are the off-diagonal elements of (77) are functions of standard deviations $(\sigma_X, \sigma_Y, \sigma_Z)$ – the diagonal elements of (77) – and covariances $(\sigma_{XY} = \sigma_{YX}, \sigma_{XZ} = \sigma_{ZX}, \sigma_{YZ} = \sigma_{ZY})$ and are given by

$$\rho_{jk} = \frac{\sigma_{jk}}{\sigma_j \sigma_k} \tag{78}$$

Where $-1 \le \rho_{jk} \le 1$.

For Mt. Ngongotaha the variance matrix $\mathbf{\Sigma}_{XYZ}$ is

$$\mathbf{\Sigma}_{XYZ} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{YX} & \sigma_Y^2 & \sigma_{YZ} \\ \sigma_{ZX} & \sigma_{ZY} & \sigma_Z^2 \end{bmatrix} = \begin{bmatrix} 1.5376E-04 & -8.8033E-06 & 1.1175E-04 \\ -8.8033E-06 & 5.9290E-05 & -6.5055E-06 \\ 1.1175E-04 & -6.5055E-06 & 9.4090E-05 \end{bmatrix} \tag{79}$$

$\mathbf{\Sigma}_{XYZ}$ is obtained from (77) using (78) and the fact that standard deviations are defined to be the positive square roots of variances.

We now wish to use these values to compute: (i) the geographical coordinates (ϕ, λ, h) and the variances and covariances of these computed quantities; and (ii) the TM Grid coordinates (E, N) and the variances and covariances of these computed quantities.

Using the Cartesian coordinates and Bowring's iterative method (Deakin & Hunter 2013) the geographical coordinates (ϕ, λ, h) related to the WGS84 ellipsoid $(a = 6378137, f = 1/298.257223563)$ are:

Mt. Ngongotaha:

$$\phi = -38^\circ\, 07'\, 06.095401''$$
$$\lambda = 176^\circ\, 11'\, 52.551149''$$
$$h = 786.1195 \text{ m}$$

With radii of curvature:

$$\rho = 6359758.0484 \text{ m (meridian)}$$
$$\nu = 6386287.4527 \text{ m (prime vertical)}$$
$$r_m = \sqrt{\rho\nu} = 6373008.9460 \text{ m (mean)}$$

The variance matrix of the computed geographical coordinates $\mathbf{\Sigma}_{\phi\lambda h}$ is given by (39) where the Jacobian matrix $\mathbf{J}^*$, given by (55) is

$$\mathbf{J}^* = \begin{bmatrix} \dfrac{-\sin\phi\cos\lambda}{\rho+h} & \dfrac{-\sin\phi\sin\lambda}{\rho+h} & \dfrac{\cos\phi}{\rho+h} \\ \dfrac{-\sin\lambda}{(\nu+h)\cos\phi} & \dfrac{\cos\lambda}{(\nu+h)\cos\phi} & 0 \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} = \begin{bmatrix} -9.6836E-08 & 6.4353E-09 & 1.2369E-07 \\ -1.3196E-08 & -1.9857E-07 & 0 \\ -7.8501E-01 & 5.2168E-02 & -6.1729E-01 \end{bmatrix} \tag{80}$$

An application of (39) gives

$$\mathbf{\Sigma}_{\phi\lambda h} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} & \sigma_{\phi h} \\ \sigma_{\lambda\phi} & \sigma_\lambda^2 & \sigma_{\lambda h} \\ \sigma_{h\phi} & \sigma_{h\lambda} & \sigma_h^2 \end{bmatrix} = \begin{bmatrix} 2.0736E-19 & -7.0431E-20 & 4.2625E-13 \\ -7.0431E-20 & 2.3184E-18 & -2.7465E-13 \\ 4.2625E-13 & -2.7465E-13 & 2.4021E-04 \end{bmatrix} \tag{81}$$

The variance matrix can be converted into an array of standard deviations/correlations

	ϕ(m)	λ(m)	h(m)
ϕ(m)	0.0029	–0.1016	0.0604
λ(m)	–0.1016	0.0077	–0.0116
h(m)	0.0604	–0.0116	0.0155

(82)

Where the standard deviations in latitude and longitude are expressed as linear quantities on the ellipsoid: $\phi(\text{m}) = \sigma_\phi \rho$, $\lambda(\text{m}) = \sigma_\lambda \nu \cos\phi$ and $h(\text{m}) = \sigma_h$. $\sigma_\phi, \sigma_\lambda, \sigma_h$ are the square-roots of the diagonal elements of (81).

The correlations in (82) are evaluated using (78) with appropriate elements of (81); for example: $\rho_{\phi\lambda} = \dfrac{\sigma_{\phi\lambda}}{\sigma_\phi \sigma_\lambda}$.

The geographical coordinates (ϕ, λ) can be transformed to TM Grid coordinates (E, N) where the central meridian is $\lambda_0 = 177^\circ$ East, the central meridian scale factor is $k_0 = 0.9996$ and the false origin offsets are $E_0 = 500{,}000$ m and $N_0 = 10{,}000{,}000$ m.

$$E = \quad 429{,}693.2527 \text{ m}$$
$$N = 5{,}780{,}748.7974 \text{ m}$$

The variance matrix of the computed grid coordinates $\mathbf{\Sigma}_{EN}$ is given by (64) where $\mathbf{\Sigma}_{\phi\lambda}$ is the upper-left block of elements from (81)

$$\mathbf{\Sigma}_{\phi\lambda} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} \\ \sigma_{\lambda\phi} & \sigma_\lambda^2 \end{bmatrix} = \begin{bmatrix} 2.0736E-19 & -7.0431E-20 \\ -7.0431E-20 & 2.3184E-18 \end{bmatrix} \tag{83}$$

And the Jacobian matrix $\mathbf{J}$, given by (69) is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial E}{\partial \phi} & \frac{\partial E}{\partial \lambda} \\ \frac{\partial N}{\partial \phi} & \frac{\partial N}{\partial \lambda} \end{bmatrix} = \frac{R}{1-\cos^2\phi\sin^2\omega}\left[\begin{array}{c|c} -\sin\phi\sin\omega & \cos\phi\cos\omega \\ \hline \cos\omega & \sin\phi\cos\phi\sin\omega \end{array}\right] = \begin{bmatrix} 55075.7002 & 5014000.4506 \\ 6373157.4857 & 43330.1054 \end{bmatrix} \tag{84}$$

where $R = r_m = \sqrt{\rho\nu}$ and $\omega = \lambda - \lambda_0$.

An application of (64) gives the variance matrix

$$\mathbf{\Sigma}_{EN} = \begin{bmatrix} \sigma_E^2 & \sigma_{EN} \\ \sigma_{EN} & \sigma_N^2 \end{bmatrix} = \begin{bmatrix} 5.8247E-05 & -1.6743E-06 \\ -1.6743E-06 & 8.3878E-06 \end{bmatrix} \tag{85}$$

The variance matrix can be converted into an array of standard deviations/correlations

	E(m)	N(m)
E(m)	0.0076	–0.0757
N(m)	–0.0757	0.0029

(86)

Systematic Errors and the Total Increment Theorem in Coordinate Transformations

The effects of *random errors* in coordinate transformation can be investigated using Propagation of Variances as outlined in sections above. To assess the effects of *systematic errors* we may use the *Total Increment Theorem* that we express as:

For a function $w = w(x, y, z, \ldots t)$ the *total increment* δw is

$$\delta w \approx \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z + \cdots + \frac{\partial w}{\partial t}\delta t \tag{87}$$

And we consider increments $\delta x, \delta y,$ etc. represent systematic errors.

Example 2

Consider the effect of 'rounding' in Cartesian to geographic transformations where it is assumed that the *X,Y,Z* Cartesian coordinates have been rounded to the nearest millimetre (mm) prior to conversion to geographic coordinates ϕ, λ, h. In this case rounding induces maximum systematic errors $\delta X = \delta Y = \delta Z = 0.0005$ m and we wish to evaluate the errors $\delta\phi, \delta\lambda, \delta h$ in latitude, longitude and height respectively.

Write ϕ, λ, h as functions of *X,Y,Z* in the general form

$$\begin{aligned} \phi &= \phi(X,Y,Z) \\ \lambda &= \lambda(X,Y,Z) \\ h &= h(X,Y,Z) \end{aligned} \tag{88}$$

And using the total increment theorem (87) write

$$\begin{aligned} \delta\phi &\approx \frac{\partial \phi}{\partial X}\delta X + \frac{\partial \phi}{\partial Y}\delta Y + \frac{\partial \phi}{\partial Z}\delta Z \\ \delta\lambda &\approx \frac{\partial \lambda}{\partial X}\delta X + \frac{\partial \lambda}{\partial Y}\delta Y + \frac{\partial \lambda}{\partial Z}\delta Z \\ \delta h &\approx \frac{\partial h}{\partial X}\delta X + \frac{\partial h}{\partial Y}\delta Y + \frac{\partial h}{\partial Z}\delta Z \end{aligned} \tag{89}$$

Equations (89) can be represented in matrix form as

$$\begin{bmatrix}\delta\phi\\ \delta\lambda\\ \delta h\end{bmatrix} \approx \begin{bmatrix}\frac{\partial\phi}{\partial X} & \frac{\partial\phi}{\partial Y} & \frac{\partial\phi}{\partial Z}\\ \frac{\partial\lambda}{\partial X} & \frac{\partial\lambda}{\partial Y} & \frac{\partial\lambda}{\partial Z}\\ \frac{\partial h}{\partial X} & \frac{\partial h}{\partial Y} & \frac{\partial h}{\partial Z}\end{bmatrix}\begin{bmatrix}\delta X\\ \delta Y\\ \delta Z\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}\delta\phi\\ \delta\lambda\\ \delta h\end{bmatrix} \approx \mathbf{J}^*\begin{bmatrix}\delta X\\ \delta Y\\ \delta Z\end{bmatrix} \tag{90}$$

And the partial derivatives in $\mathbf{J}^*$ are given in (55)

If we round the Cartesian coordinates of Mt. Ngongotaha to the nearest mm and assume that the maximum rounding errors are $\delta X = \delta Y = \delta Z = 0.0005$ m then using (80) and (90)

$$\begin{bmatrix}\delta\phi\\ \delta\lambda\\ \delta h\end{bmatrix} \approx \mathbf{J}^*\begin{bmatrix}\delta X\\ \delta Y\\ \delta Z\end{bmatrix} = \begin{bmatrix}0.000003\\ -0.000022\\ -0.000675\end{bmatrix}\begin{matrix}\text{seconds}\\ \text{seconds}\\ \text{metres}\end{matrix}$$

So we can assume that (in this case) rounding to the nearest mm before transforming Cartesian to Geographic will not induce errors greater than 0.00005 seconds of arc in latitude or longitude or errors greater than 0.001 mm in ellipsoidal height.

Example 3

Consider the effect of rounding in the transformation from Geographic coordinates (ϕ,λ) to TM Grid (E,N).

Similarly to the example above we may write

$$\begin{aligned} E &= E(\phi,\lambda)\\ N &= N(\phi,\lambda)\end{aligned} \tag{91}$$

And using the total increment theorem (87) write

$$\begin{aligned}\delta E &\approx \frac{\partial E}{\partial\phi}\delta\phi + \frac{\partial E}{\partial\lambda}\delta\lambda\\ \delta N &\approx \frac{\partial N}{\partial\phi}\delta\phi + \frac{\partial N}{\partial\lambda}\delta\lambda\end{aligned} \tag{92}$$

Equations (92) can be represented in matrix form as

$$\begin{bmatrix}\delta E\\ \delta N\end{bmatrix} \approx \begin{bmatrix}\frac{\partial E}{\partial\phi} & \frac{\partial E}{\partial\lambda}\\ \frac{\partial N}{\partial\phi} & \frac{\partial N}{\partial\lambda}\end{bmatrix}\begin{bmatrix}\delta\phi\\ \delta\lambda\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}\delta E\\ \delta N\end{bmatrix} \approx \mathbf{J}\begin{bmatrix}\delta\phi\\ \delta\lambda\end{bmatrix} \tag{93}$$

And the partial derivatives in **J** are given in (69)

Now suppose that ϕ,λ for Mt Ngongotaha are rounded to the nearest 0.0001″ before transforming to *E,N* grid coordinates. This supposes maximum rounding errors $\delta\phi = \delta\lambda = 0.00005''$. Then using (84) and (93)

$$\begin{bmatrix}\delta E\\ \delta N\end{bmatrix} \approx \mathbf{J}\begin{bmatrix}\delta\phi\\ \delta\lambda\end{bmatrix} = \begin{bmatrix}0.0012\\ 0.0016\end{bmatrix}\begin{matrix}\text{metres}\\ \text{metres}\end{matrix}$$

So we can assume that (in this case) rounding to the nearest 0.0001″ before transforming Geographic to TM Grid will not induce errors greater than 0.002 metres.

References

Deakin, R.E., 2004, *The Standard and Abridged Molodensky Coordinate Transformation Formulae,* Lecture Notes, School of Mathematical & Geospatial Sciences, RMIT University, Melbourne, April 2004, 22 pages.

Deakin, R.E. and Hunter, M.N., 2013, *Geometric Geodesy – Part A,* Lecture Notes, School of Mathematical & Geospatial Sciences, RMIT University, Melbourne, 3rd printing, January 2013, v+145 pages.

Deakin, R.E., Hunter, M.N. and Karney, C.F.F., 2012, 'A Fresh Look at the UTM Projection: Karney-Krueger equations', presented at the Surveying and Spatial Sciences Institute (SSSI) Land Surveying Commission National Conference, Melbourne, 18-21 April, 2012. http://www.academia.edu/7826254/A_Fresh_Look_at_the_UTM_Projection_Karney-Krueger_equations (accessed 29-Oct-2014)

Harvey, B.R., 1986, 'Transformation of 3D co-ordinates', *The Australian Surveyor,* Vol. 33, No. 2, pp. 105-125, June 1986. http://www.surveying.unsw.edu.au/ls/Trans_3D_Aust_Surv_86.pdf (accessed 29-Oct-2014)

Mikhail, E.M., 1976, *Observations and Least Squares,* IEP—A Dun-Donnelley, New York.

Soler, T., 1976, *On Differential Transformations Between Cartesian and Curvilinear (Geodetic) Coordinates,* Report No. 236, Reports of the Department of Geodetic Science, The Ohio State University, Columbus, Ohio 43212 USA, 82 pages, January 1976. http://www.academia.edu/1223237/On_differential_transformations_between_Cartesian_and_curvilinear_geodetic_coordinates (accessed 29-Oct-2014)